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THE APPROXIMATE SUMMATION OF SERIES, IN WHICH EACH TERM IS A FUNCTION OF THE CORRESPONDING TERM OF AN ARITHMETICAL PROGRESSION.

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Harmonical Progressions.

The height of a column of air of unit cross section may be shown, by the integral calculus or otherwise, to be equal to K. $\log \frac{P}{P'}$, where P is the pressure at the bottom, P' the pressure at the top, and K a constant. But the same height may be expressed as the sum of the heights of n short columns of air of equal mass. If the weight of each is w the pressures at the centres of the sections will be approximately $P-\frac{1}{2}w$, $P-\frac{3}{2}w$, etc., and by Boyle's Law their heights will be :—

$$\frac{C}{P-\frac{1}{2}w}, \quad \frac{C}{P-\frac{3}{2}w}, \quad \ldots\ldots \quad \frac{C}{P-(n-\frac{1}{2})w},$$

where C is another constant. The sum of these heights is the total height.

$$\therefore \frac{C}{P-\frac{1}{2}w} + \frac{C}{P-\frac{3}{2}w} + \ldots + \frac{C}{P-(n-\frac{1}{2})w} = \text{K} . \log \frac{P}{P-nw}$$

$$\text{or, } \frac{1}{P-\frac{1}{2}w} + \frac{1}{P-\frac{3}{2}w} + \ldots + \frac{1}{P-(n-\frac{1}{2})w} = \frac{\text{K}}{\text{C}} . \log \frac{P}{P-nw}$$

Putting $n = 1$ we have :—

$$\frac{1}{P-\frac{1}{2}w} = \frac{K}{C} . \log \frac{P}{P-w}$$

$$\therefore \quad \frac{K}{C} = \frac{1}{P-\frac{1}{2}w} . \frac{1}{\log \frac{P}{P-w}}$$

$$\therefore \frac{1}{P-\frac{1}{2}w} + \frac{1}{P-\frac{3}{2}w} + \ldots + \frac{1}{P-(n-\frac{1}{2})w}$$

$$= \frac{1}{P-\frac{1}{2}w} . \frac{\log \frac{P}{P-nw}}{\log \frac{P}{P-w}}$$

Putting $P - (n - \frac{1}{2}) w = A$, and writing the series the other way about :—

$$\frac{1}{A} + \frac{1}{A+w} + \ldots + \frac{1}{A+(n-1)w}$$

$$= \frac{\dfrac{A+(n-\frac{1}{2})w}{A-\frac{1}{2}w}}{\{A+(n-1)w\} . \log \dfrac{A+(n-\frac{1}{2})w}{A+(n-\frac{3}{2})w}} \qquad \text{FORMULA I.}$$

The best results are obtained when A is much greater than w. An empirical variation of Formula I was obtained as follows :— The denominator of the above was observed to equal

$$w . \log \left(\frac{A+(n-\frac{1}{2})w}{A+(n-\frac{3}{2})w}\right)^{(A+[n-1]w)/w}$$

$$= w . \log \left(1 + \frac{w}{A+(n-\frac{3}{2})w}\right)^{(A/w+n-1)}$$

$$= w . \log \left(1 + \frac{1}{A/w+(n-\frac{3}{2})}\right)^{(A/w+n-1)}$$

which is the limit, when n is infinite,

$= w. \log e.$

$= w$, if hyperbolic logarithms are used.

The expression then becomes :—

$$\frac{1}{w} \log_e \frac{A+(n-\frac{1}{2})w}{A-\frac{1}{2}w}$$

Then, as a first approximation, we have :—

$$\frac{1}{A} + \frac{1}{A+w} + \ldots + \frac{1}{A+(n-1)w}$$

$$= \frac{1}{w} . \log_e \frac{A+(n-\frac{1}{2})w}{A-\frac{1}{2}w}$$

$$= \frac{1}{w} . \log_e \left(1 + \frac{nw}{A-\frac{1}{2}w}\right)$$

Putting $n = 1$,

$$\frac{1}{A} = \frac{1}{w} \cdot \log_e \left(1 + \frac{w}{A - \frac{1}{2} w}\right)$$

$$\therefore \quad \frac{w}{A - \frac{1}{2} w} = e^{w/A} - 1 \,.$$

Substituting this approximate identity in the equation above we obtain the following formula which holds even when n is not large :—

$$\frac{1}{A} + \frac{1}{A + w} + \ldots + \frac{1}{A + (n - 1) w}$$

$$= \frac{1}{w} \cdot \log_e \left\{1 + n \left(e^{w/A} - 1\right)\right\} \quad \text{Formula II.}$$

EXAMPLES :

Series.	True Sum.	Formula I.	Approx. Error in parts per 100,000	Formula II.	Approx. Error in parts per 100,000
$\frac{1}{2} + \frac{1}{3}$	·83333	·83939	+727	·83180	−184
$\frac{1}{2} + \cdots + \frac{1}{10}$	1·92897	1·94429	+795	1·92257	−332
$\frac{1}{2} + \cdots + \frac{1}{25}$	2·81596	2·83284	+601	2·80755	−299
$\frac{1}{10} + \frac{1}{11} + \frac{1}{12}$	·27424	·2 428	+15	·27423	−4
$\frac{1}{10} + \cdots + \frac{1}{20}$	·76877	·76897	+26	·76866	−14
$\frac{1}{10} + \cdots + \frac{1}{25}$	·98699	·9872	+26	·98683	−16
$\frac{1}{100} + \frac{1}{101}$	·0199010	·0199010	?	·0199010	?
$\frac{1}{100} + \cdots + \frac{1}{115}$	·14911	·14911	?	·14911	?
$\frac{1}{100} + \cdots + \frac{1}{150}$	·41380	·41380	?	·41380	?

It may be observed that the summation of the series $\frac{1}{A} + \frac{1}{A+w} + \ldots + \frac{1}{A+(n-1)w}$ can be carried out to any required degree of accuracy by an application of one of Euler's asymptotic series. Euler states (Inst. Calc. Diff., 1755, Pars Posterior, cap. VI) that

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{x} = \gamma + \log x + \frac{1}{2x} - \frac{B_1}{2x^2} + \frac{B_2}{4x^4} - \frac{B_3}{6x^6} + \ldots$$

where γ = constant, and B_1, B_2, B_3, ... are Bernoulli's numbers, viz., $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, ... An adaptation of the above formula gives:—

$$\frac{1}{A} + \frac{1}{A+w} + \ldots + \frac{1}{A+(n-1)w} = \frac{1}{w} \cdot \log \frac{A+(n-1)w}{A-w} + \frac{1}{2(A+[n-1]w)} - \frac{1}{2(A-w)} - \frac{B_1 w}{2(A+[n-1]w)^2} + \frac{B_1 w}{2(A-w)^2} + \ldots$$

The alternation of + and − signs makes it necessary to carry the series on the right to five terms, when the summation is usually accurate to several places of decimals.

Relations between the powers of e.

If, in Formula II, w is put = 1 we get:—

$$\frac{1}{A} + \frac{1}{A+1} + \ldots + \frac{1}{A+n-1} = \log_e \left\{ 1 + n\left(e^{1/A} - 1\right) \right\}$$

or $e^{\left(\frac{1}{A} + \frac{1}{A+1} + \ldots + \frac{1}{A+n-1}\right)} = 1 + n(e^{1/A} - 1) = n.e^{1/A} - n + 1$,

from which the following approximate relations are obtained:—

			True Values.	
	A.	B.	A.	B.
If $n = 2$, $A = 1$	e^{1+1}	$= 2e - 1$	4·482	4·437
,, $A = 2$	$e^{\frac{1}{2}+\frac{1}{3}}$	$= 2e^{\frac{1}{2}} - 1$	2·301	2·297
,, $A = 3$	$e^{\frac{1}{3}+\frac{1}{4}}$	$= 2e^{\frac{1}{3}} - 1$	1·792	1·791
If $n = 3$, $A = 1$	$e^{1+\frac{1}{2}+\frac{1}{3}}$	$= 3e - 2$	6·255	6·155
,, $A = 2$	$e^{\frac{1}{2}+1+\frac{1}{4}}$	$= 3e^{\frac{1}{2}} - 2$	2·955	2·946
,, $A = 3$	$e_{\frac{1}{3}}+\frac{1}{4}+1$	$3e^{\frac{1}{3}} - 2$	2·189	2·187

and so *ad infinitum*.

Arithmetical Progressions and Cognate Series.

By a method analogous to that used for Formula I the summation of other series can be effected. Instead of taking PV constant, as in the case of Boyle's Law, assume the theoretical relation $P^{-m}V \equiv$ constant. Then the heights, which are proportional to the volumes, may be integrated thus:—

$$C(P-\tfrac{1}{2}w)^m + C(P-\tfrac{3}{2}w)^m + \dots + C(P-[n-\tfrac{1}{2}]w)^m$$

$$= \int_{P'}^{P} K \,.\, P^m dP$$

$$= \frac{K}{m+1} \cdot \left(P^{(m+1)} - P'^{(m+1)}\right)$$

$$= \frac{K}{m+1} \cdot \left(P^{(m+1)} - [P - nw]^{(m+1)}\right)$$

whence, by a process similar to that employed for the harmonical progression, we get:—

$$A^m + (A+w)^m + \dots + (A+[n-1]w)^m$$

$$= (A+[n-1]w)^m \cdot \frac{\left(\dfrac{A-\frac{1}{2}w}{A+(n-\frac{1}{2})w}\right)^{m+1} - 1}{\left(\dfrac{A+(n-\frac{3}{2})w}{A+(n-\frac{1}{2})w}\right)^{m+1} - 1} \qquad \text{FORMULA III.}$$

If m is put $= 1$ the above formula reduces to:—

$$A + (A+w) + \dots + (A+[n-1]w) = \tfrac{1}{2}n(2A+[n-1]w),$$

which is the usual formula for the exact summation of an arithmetical progression. If $m = 0$ each term becomes unity, and the sum is n, exactly. But in other cases the summation is approximate only. Those series which are convergent may be summed to infinity; the general formula, derived from III, being:—

$$\frac{1}{A^m} + \frac{1}{(A+w)^m} + \dots \text{ ad inf.} = \frac{1}{(m-1)\,.\,w\,.\,(A-\frac{1}{2}w)^{(m-1)}}$$

EXAMPLES:

Series.	True Sum.	Sum by Formula III.	Approx. Error in parts per 100,000.
$\frac{1}{10^2}+\frac{1}{11^2}+\frac{1}{12^2}$	·0252089	·0252193	+ 41
$\frac{1}{10^2}+\frac{1}{11^2}+\ldots$ ad inf.	·105166	·105263	+ 92
$\frac{1}{\sqrt{10}}+\frac{1}{\sqrt{11}}+\frac{1}{\sqrt{12}}$	·906414	·906455	+ 5
$\frac{1}{\sqrt[3]{10}}+\frac{1}{\sqrt[3]{11}}+\frac{1}{\sqrt[3]{12}}$	1·35059	1·35063	+ 3
$10^2+11^2+12^2$	365·	365·039	+ 11
$10^3+11^3+12^3$	4059·	4060·20	+ 30
$\sqrt{10}+\sqrt{11}+\sqrt{12}$	9·94300	9·94284	− $1\frac{1}{2}$
$\sqrt[3]{10}+\sqrt[3]{11}+\sqrt[3]{12}$	6·66784	6·66775	− $1\frac{1}{3}$
$\sqrt{100}+\sqrt{101}+\ldots+\sqrt{115}$	165·8534	165·8516	− 1

Derivatives of Formula II.

Formulæ for the approximate summation of series of the form $\frac{1}{A^m}+\frac{1}{(A+w)^m}+\ldots$ can also be obtained by repeated differentiation of the empirical Formula II. E.g. :—

$$\frac{1}{A}+\frac{1}{A+w}+\ldots+\frac{1}{A+(n-1)w}=\frac{1}{w}.\log_e\left\{1+n\left(e^{w/A}-1\right)\right\}$$

$$\therefore\ \frac{-1}{A^2}+\frac{-1}{(A+w)^2}+\ldots+\frac{-1}{(A+[n-1]w)^2}=\frac{1}{w}.\frac{n.e^{w/A}(-w/A^2)}{1+n(e^{w/A}-1)}$$

$$\therefore\ \frac{1}{A^2}+\frac{1}{(A+w)^2}+\ldots+\frac{1}{(A+[n-1]w)^2}$$

$$=\frac{1}{A^2}.\frac{n.e^{w/A}}{1+n(e^{w/A}-1)}\qquad \text{FORMULA IV.}$$

And in the special case where n is infinite,

$$\frac{1}{A^2} + \frac{1}{(A+w)^2} + \ldots \text{ ad inf.} = \frac{1}{A^2} \cdot \frac{e^{w/A}}{e^{w/A}-1}$$

EXAMPLES:

Series.	True Sum.	Sum by Formula III.	Error in parts per 100,000.	Sum by Formula IV.	Error in parts per 100,000.
$\frac{1}{10^2} + \frac{1}{11^2} + \frac{1}{12^2}$	·0252089	·0252193	+41	·0252032	–23
$\frac{1}{10^2} + \frac{1}{11^2} + \ldots$ ad inf.	·105166	·105263	+92	·105083	–79

Other Series.

By a modification of the foregoing methods approximate summations can be found for geometrical progressions, and also such trigonometrical series as $\sin A + \sin(A+w) + \ldots + \sin(A+nw)$, but as these admit of exact summation the results are of no practical interest.

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